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A NOTE ON THE PERTURBATIONS OF SINGULAR VALUES.(U)

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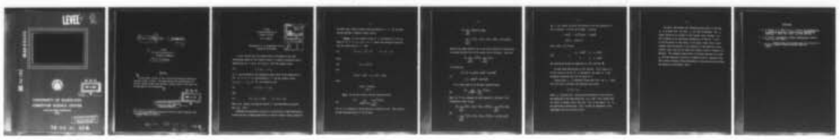

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A Note on the Perturbations of Singular Values

(10) G. W. Stewart*

Abstract

In this note a variant of the classical perturbation theorem for singular values is given. The bound explain why perturbations will tend to increase rather than decrease singular values of the same order of magnitude as the perturbation.

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A Note on the Perturbation of Singular Values

G. W. Stewart

Dedicated to A. S. Householder on his seventy-fifth birthday

In this note we shall be concerned with a sharpening of the usual perturbation bounds for the singular values of a general rectangular matrix. Specifically let X be an $n \times p$ matrix. Then the singular values

(1) $$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p$$

of X may be defined as the nonnegative square roots of the eigenvalues of X^TX. If $\tilde{X} = X + E$ is a perturbation of X and the singular values $\tilde{\sigma}_1 \geq \tilde{\sigma}_2, \geq \ldots \geq \tilde{\sigma}_p$ of X^TX are ordered so that

(2) $$\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \ldots \geq \tilde{\sigma}_p,$$

then

(3) $$|\sigma_i - \tilde{\sigma}_i| \leq \|E\| \qquad (i = 1,2,\ldots,p),$$

where $\|E\|$ denotes the spectral norm of E (for definitions and proofs see, e.g., [2]).

Although the perturbation bound (3) is satisfactory in most applications, it does not give a complete description of how the singular values, especially

the small ones, actually behave under perturbations in X. The following theorem provides a somewhat clearer picture.

<u>Theorem</u>. Let the singular values of X be ordered as in (1) and those of $\tilde{X} = X + E$ as in (2). Let P denote the orthogonal projection onto the column space of X. Then

$$(4) \qquad \tilde{\sigma}_i^2 = (\sigma_i + \xi_i)^2 + \eta_i^2 \qquad (i = 1,2,\ldots,p)$$

where

$$(5) \qquad |\xi_i| \le \| P E \|$$

and

$$\inf\left[(I - P)E\right] \le \eta_i \le \|(I - P)E\|.$$

Here

$$\inf(E) = \inf_{\|x\|=1} \|Ex\|.$$

<u>Proof</u>. We use the classical min-max characterization

$$(6) \qquad \sigma_i^2 = \min_{\dim(\mathcal{X})=p-i+1} \; \max_{\substack{x\in\mathcal{X} \\ \|x\|=1}} x^T(X^TX)x.$$

Let $\mathcal{X}$ be a subspace for which equality is attained in (6). Then from the min-max characterization of $\tilde{\sigma}_i^2$, we have

$$\tilde{\sigma}_i^2 \le \max_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^T\left[(X+E)^T(X+E)\right]x$$

$$= \max_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^T\left[X^TX + E^TPX + X^TPE + E^TP^2E + E^T(I-P)^2E\right]x,$$

where we have made extensive use of the usual properties of projections in passing from the first to the second form of the bound. Now since

$$\sigma_i^2 = \max_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^T(X^TX)x = \max_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} \|Xx\|,$$

it follows that

$$\tilde{\sigma}_i^2 \le \sigma_i^2 + 2\sigma_i\|PE\| + \|PE\|^2 + \|(I-P)E\|^2$$

(7)

$$= (\sigma_i + \|PE\|)^2 + \|(I-P)E\|^2$$

For a lower bound we use the dual characterization

(8) $$\sigma_i^2 = \max_{\dim(\mathcal{X})=i} \min_{\substack{x \varepsilon \mathcal{X} \\ x=1}} x^T(X^TX)x.$$

Again let $\mathcal{X}$ be a subspace for which equality is attained in (8). Proceeding as above, we get

$$\tilde{\sigma}_i^2 \ge \min_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^T\left[X^TX + E^TPX + X^TPE + E^TP^2E\right]x + \min_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^TE^T(I-P^2)Ex$$

(9)

$$\ge \min_{\substack{x \varepsilon \mathcal{X} \\ \|x\|=1}} x^T\left[X^TX + E^TPX + X^TPE + E^TP^2E\right]x + \inf\left[(I-P)E\right]^2.$$

Let x be a vector for which the minimum in the last expression of (9) is attained. Calling this minimum μ, we have

$$\mu = \|Xx\|^2 + 2(x^T X^T)(PEx) + \|PEx\|^2 .$$

$$\geq (\|Xx\| - \|PEx\|)^2.$$

Since $\|Xx\| \geq \sigma_i^2$, we have

$$(10) \qquad \mu \geq \begin{cases} (\sigma_i - \|PE\|)^2 & \text{if} \quad \sigma_i \geq \|PE\| \\ 0 & \text{if} \quad \sigma_i \leq \|PE\| \end{cases} .$$

The theorem now follows on combining (7), (9), and (10). ■

We make three observations on this theorem. First, there is a trivial variant in which PE is replaced by ER, where R is the orthogonal projection onto the row space of X.

Second, when σ_i is reasonably larger than $\|E\|$, say $\sigma_i > 5\|E\|$, the first term in the bound (4) dominates and we have

$$\tilde{\sigma}_i \doteq \sigma_i + \xi_i,$$

where ξ_i satisfies (5). The classical perturbation result cited at the beginning of this note would give $|\xi_i| \leq \|E\|$. Since $\|PE\| \leq \|E\|$ our result is sharper; and in fact when $n >> p$ we may expect PE to be significantly smaller than $\|E\|$, so that (5) represents a true improvement over the classical result.

The third and perhaps most interesting observation is that when σ_i is of order $\|E\|$, the term η_i will tend to dominate. Now η_i always represents an increase in the singular value, and when $n \gg p$ this increase can be significant, depending as it does on $(I-P)E$. To put the matter in other words, if one takes a matrix with a small singular value and perturbs it by quantities of the same size as that singular value, then one can expect the singular value to increase, not decrease. This tendency toward better conditioned matrices with larger σ_p has been observed in practice in connection with a regression problem in which simulated random perturbations in the data seriously biased the regression coefficients [1,3].

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